Cardano's method for solving cubic equations

Suitable for: top sets in year 11 upwards. Must have experience of multiplying out brackets and quadratic formula or completing the square.

Worked example for equations of the form $x^3 + px + q = 0$.

Solve: $\quad x^3 + 15x + 8 = 0 \qquad (*)$

Comparing (*) with the identity $(u+v)^3 - 3uv(u+v) - (u^3+v^3) \equiv 0$, if we can find u and v that satisfy

$$-3uv = 15 \quad (1) \quad \text{and} \quad -(u^3+v^3) = 8 \quad (2)$$

then we will have a solution to (*). Now (1) and (2) give

$$u^6 + 8u^3 - 125 = 0$$

which has solutions $u^3 = -4 \pm \sqrt{141}$; (2) then gives $v^3 = -4 \mp \sqrt{141}$.

So $x = \sqrt[3]{-4+\sqrt{141}} + \sqrt[3]{-4-\sqrt{141}}$ is a solution (in fact the only solution) to (*).

Equations of the form $x^3 + ax^2 + bx + c = 0$. $\qquad (**)$

The trick here is to transform the equation to one of the form which we have just solved (this was Cardano's breakthrough - del Ferro and Tartaglia had previously used the method above). If we substitute $x = y - a/3$ into (**), then the resulting cubic (in y) has no y^2 term and can be solved in the same way as (*).

E.g. $\quad x^3 + 6x^2 - 3x + 6 = 0 \quad (***)$

Let $x = y - 2$, and (***) becomes:

$(y-2)^3 + 6(y-2)^2 - 3(y-2) + 6 = 0$, which reduces to

$y^3 - 15y + 28 = 0$. This has solution $y = \sqrt[3]{-14+\sqrt{71}} + \sqrt[3]{-14-\sqrt{71}}$.

So (***) has solution $x = -2 + \sqrt[3]{-14+\sqrt{71}} + \sqrt[3]{-14-\sqrt{71}}$.

Further examples and tasks for pupils

Applying the method to $x^3+6x-20=0$ gives $x=\sqrt[3]{10+\sqrt{108}}+\sqrt[3]{10-\sqrt{108}}$. An exercise for good pupils would be to demonstrate that this is in fact the number 2.

The equation $x^3-15x-4=0$ gives $x=\sqrt[3]{2+\sqrt{-121}}+\sqrt[3]{2-\sqrt{-121}}$. This involves finding the square root of a negative number, which of course cannot be done. However it is easily shown that the original cubic has three *real* roots - further mathematicians could show that this expression is in fact equal to 4. The historical motivation for complex numbers comes from this sort of problem and not from quadratics with negative discriminant.

It's useful to stress the advantage of the method above over the other techniques that pupils might use to solve an equation like (*) - e.g. trial and improvement or an iterative solution.

Good pupils could be encouraged to find a 'cubic formula' for equations of the form $x^3+px+q=0$. Another extension (particularly for those who've studied some complex number work) is to establish how other roots of a cubic expression are found.

The website www.york.ac.uk/depts/maths/histstat/cubic.pdf has details of a method for solving the general quartic, for any pupils interested. This will be extremely tricky even for the very best school mathematicians. There is no solution to the general quintic equation.

Further reading:

Birkhoff, G. and MacLane, S. *A Survey of Modern Algebra.* MacMillan, 1941.
Cuoco, Al. *Mathematical Connections.* MAA, 2005.
Dunham, William. *Journey Through Genius.* Wiley, 1990.