Continued Fractions

(Ideas could be used with top sets in most years – some knowledge of solving quadratics would be helpful)

What is the value of the following expression?

$$1+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\ldots}}}}$$

There are several ways to get to the correct answer. One is to work out the values of

$$1+\frac{1}{2}, \qquad 1+\cfrac{1}{2+\cfrac{1}{2}}, \qquad 1+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2}}}, \qquad \text{etc.}$$

This sequence gives: $3/2,\ 7/5,\ 17/12,\ 41/29,\ 99/70,\ 239/169,\ldots$ (*)

or, as decimals, 1.5, 1.4, 1.4167, 1.4138, 1.4143, 1.4142,…

It should become clear that this is an oscillating, convergent sequence, and it's not too much of a leap to suggest that the answer to the original question might be $\sqrt{2}$.

An alternative, particularly for those familiar with quadratics, is to notice that if we call the required value *x*, then *x* must satisfy the equation $x = 1+\frac{1}{1+x}$. Solving this gives $x = \sqrt{2}$ or $-\sqrt{2}$, but the required answer must be positive, so our answer is again $\sqrt{2}$.

The expression at the beginning is the *continued fraction* for $\sqrt{2}$, and is sometimes written as [1, 2, 2, 2, 2,…]. Even though $\sqrt{2}$ is irrational, it has a continued fraction with a recurring pattern. All quadratic surds in fact have a recurring continued fraction, and all rational numbers have a terminating continued fraction.

There is an easy algorithm for finding the continued fraction of a number using a calculator:

1. Enter the number;
2. Write down the integer part of the number;
3. Take away the integer part and then reciprocate;
4. Use this answer to start again at instruction 2.

At some point this might fail as calculators only store values to a certain accuracy. It would be useful for pupils to now calculate some continued fractions, say for *e*, π , $\sqrt{3}$ (the last of these could be tackled without a calculator by pupils who have done some rationalising of denominators).

The continued fraction for π is [3, 7, 15, 1, 292, 1, 1,…] (with no obvious pattern), and calculating the fraction corresponding to [3, 7, 15, 1] gives $^{335}/_{113}$, an extremely good approximation to π. Various continued fractions connected to *e* give very clear patterns.

The fact that continued fractions give good approximations to irrational numbers allows us to find integer solutions to various instances of **Pell's Equation**. This is an equation of the form:

$$x^2 - Dy^2 = 1,$$

Where *D* is not a square (If *D* is a square there are no solutions which is easily seen by factorising the left hand side). The equation can be re-arranged as follows:

$$\frac{x}{y} = \sqrt{D + \frac{1}{y^2}}.$$

Now if *y* is large, this boils down to $\frac{x}{y} \approx \sqrt{D}$. So if we wish to solve the specific equation

$$x^2 - 2y^2 = 1, \qquad (**)$$

we might try to find values of *x* and *y* so that $\frac{x}{y} \approx \sqrt{2}$. But this is exactly what we did when we calculated the sequence of fractions (*). When we try the first of these ($x = 3$, $y = 2$) we see that we get a solution to (**), but with the second pair we do not (we get -1 instead). If we carry on, we see that every other fraction in the list (*) gives us a solution to (**), so for example $99^2 - 2 \times 70^2 = 1$.

Questions

1. What is the exact value of [1, 1, 1, 1, 1,…]? Work out the first few fractions – what do you notice?
2. Find the continued fraction for $\sqrt{5}$. Use your answer to find three pairs *x*, *y* of positive integer solutions to the equation $x^2 - 5y^2 = -1$.
3. Use Excel to find the continued fractions for *e*, $\sqrt{e}$, $\sqrt[3]{e}$, $\sqrt[4]{e}$ (after a while the continued fraction given will not be valid, but you should be able to establish a pattern in each case before this happens). Predict the continued fraction for $\sqrt[5]{e}$ and check your prediction.
4. [Hard]. Find three positive integer pairs *x*, *y* which satisfy the equation $x^2 - y^2 - xy = 1$. Explain how you could find as many as you pleased. [Show by induction why this works].