

Second order homogeneous linear differential equations with constant coefficients: an alternative approach to solution.

The aim of this note is to outline a method of solution for equations of the form:

$$\frac{d^2 y}{dx^2} + b \frac{dy}{dx} + cy = 0 \quad (*)$$

and to show how one can derive the standard 'recipe' for their solution. Most A level and undergraduate texts give an explanation on one of the following lines:

- If the auxiliary equation $\lambda^2 + b\lambda + c = 0$ has real distinct roots k_1 and k_2 , then $y = Ae^{k_1 x} + Be^{k_2 x}$ satisfies the equation (*), has the 'right' number of arbitrary constants and therefore is the general solution. For complex roots, use the same solution and appeal to the identity $e^{i\theta} = \cos\theta + i\sin\theta$ to derive the general solution, with inevitable questions about the nature of the arbitrary constants – real or complex? In the case of a repeated root, k , notice that there aren't enough arbitrary constants, but $y = (Ax + B)e^{kx}$ works, and does have the required number.
- Introduce a differential operator D and write (*) in the form $(D^2 + bD + c)(y) = 0$. After justifying that differential operators commute, the different cases can be established, again with an appeal to Euler's formula.

The first of these methods is unsatisfactory, as it is rarely accompanied by a justification (at least in A level texts) of the necessity and sufficiency of two arbitrary constants for a second order differential equation. The second method is better, as it can be seen that the constants arise from two successive integrations, but I have found that A level further mathematicians prefer the technique outlined below. It has the added advantage, I feel, of requiring no mention of complex numbers, at least until one derives the standard recipe.

Method for solution of (*)

Use the substitution $y = ze^{kx}$ to transform the equation to:

$$e^{kx} \left(\frac{d^2 z}{dx^2} + (2k + b) \frac{dz}{dx} + (k^2 + bk + c)z \right) = 0.$$

Choosing $k = -\frac{b}{2}$ reduces this equation to

$$\frac{d^2 z}{dx^2} = \frac{b^2 - 4c}{4} z \quad (**)$$

Now the equation (**) will take one of three forms, according to whether the discriminant of the auxiliary quadratic is positive, zero or negative. So (**) can be written as one of

$$\frac{d^2 z}{dx^2} = \omega^2 z \quad (1)$$

$$\frac{d^2 z}{dx^2} = 0 \quad (2)$$

$$\frac{d^2 z}{dx^2} = -\omega^2 z \quad (3).$$

Case (2) is most easily solved, integrating twice to give $z = Ax + B$, and so $y = e^{kx}(Ax + B)$. The remaining two cases can be dealt with by setting $v = \frac{dz}{dx}$,

giving $\frac{d^2 z}{dx^2} = \frac{dv}{dx} = \frac{dz}{dx} \frac{dv}{dz} = v \frac{dv}{dz}$. Then, for example, equation (1) becomes

$$v \frac{dv}{dz} = \omega^2 z$$

$$\int v dv = \int \omega^2 z dz$$

$$v^2 = \omega^2 z^2 + c \quad \text{for some arbitrary constant } c. \text{ Thus}$$

$$\frac{dz}{dx} = \sqrt{\omega^2 z^2 + c}$$

$$\int \frac{1}{\sqrt{\omega^2 z^2 + c}} dz = \int dx. \text{ The integral on the left is found by using one}$$

of the substitutions $u = \frac{\sqrt{-c}}{\omega} \sinh z$ or $u = \frac{\sqrt{c}}{\omega} \cosh z$, depending on the sign of c ,

giving $\frac{1}{\omega} \ln(\omega z + \sqrt{\omega^2 z^2 + c}) = x + d$ for some arbitrary constant d .

Making z the subject gives $z = Ae^{\omega x} + Be^{-\omega x}$, for some constants A and B (depending on the constants of integration c and d), and so $y = Ae^{(k+\omega)x} + Be^{(k-\omega)x}$. But $k+\omega$ and $k-\omega$ are the roots of the auxiliary equation, giving the recipe result for real distinct roots. Case (3) is solved similarly and it is left as an exercise to derive the recipe result for complex roots.